Examinee’s No. ____________________

Department of Nuclear Engineering
and Management, The University of Tokyo
2010 Entrance Examination for the Master Course

Mathematics

August 31, 2009
13:00～15:30

Notice

1. Do not open this booklet until the start of the examination.

2. Write your examinee’s number on top of this sheet. Do not write your examinee’s number or your name anywhere else in this booklet.

3. Write the outline of your solution below the question and put only the answer in the box.

4. Sheets for calculation are distributed separately.

5. There are 20 questions in total. Choose and answer 15 questions out of 20.
Check the number of the questions you have selected in the bottom row.
Notice that you cannot answer more than 15 questions.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

Q1.

Region D is bounded by the line $y = x + 2$ and the parabola $y = x^2$.

(1) Obtain the area of D.

(2) Obtain the centroid (center of mass) of D.

Q2.

Positive integers are arranged as shown below. What is the number at the cell of $I = 101$ and $J = 50$?

	$I = 1$	2	3	4	·
$J = 1$	1	2	9	10	·
2	4	3	8	11	·
3	5	6	7	12	·
4	16	15	14	13	·
·	·	·	·	·	·

Q3.

Three real numbers 1, A and B $(A < B)$ can be ordered to form a geometric sequence and an arithmetic sequence. Find all pairs of A and B.

Q4.

Obtain the center of the circle passing through three points in Cartesian coordinates, $\mathrm{A}(1,-2,1)$, $\mathrm{B}(3,1,7)$ and $\mathrm{C}(2,0,6)$.

Q5.

Find the value of Z in the following calculation.

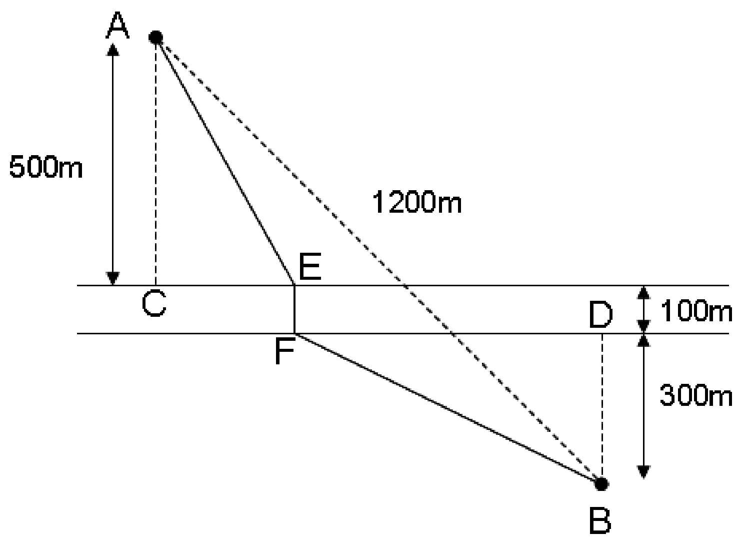
$$
\begin{array}{rrr}
 & \square & \square \\
\times & \square & \square \\
\hline
 & 2 & \square \\
6 & \square & \\
\hline
7 & Z & \square
\end{array}
$$

Q6.

When a three-digit number in nonary notation (base-9 number system) is expressed in septenary notation (base-7 number system), the order of the digits is reversed. Show this number in octal notation (base-8 number system).

Q7.

The following diagram shows that the width of the river, which lies between points A and B, is 100 m. It also shows that the distances between points A and C, between points B and D, and between points A and B are 500 m, 300 m, and 1200 m, respectively. A bridge EF orthogonal to the river should be constructed such that the route AEFB is the shortest. Find the distance between points C and E.



Q8.

Obtain the eigenvalues and the corresponding eigenvectors of a matrix

$$X = \begin{pmatrix} 1 & x & x & x \\ x & 1 & x & x \\ x & x & 1 & x \\ x & x & x & 1 \end{pmatrix},$$

where $x > 0$.

Q9.

Obtain the function $y(x)$ which satisfies the following differential equation

$$\frac{d^2y}{dx^2} = 2y^3 + 2y \,,$$

under the boundary conditions that $y(0) = 0$ and $\left.\frac{dy}{dx}\right|_{x=0} = 1$.

Q10.

Obtain the volume of the solid body which is expressed by the following inequality

$$|x+y+z|+|-x+y+z|+|x-y+z|+|x+y-z|\leq 4,$$

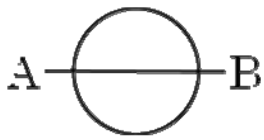
where x, y and z denote the Cartesian coordinates.

Q11.

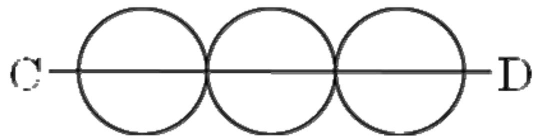
(1) In how many ways can the figure (a) be drawn with a single stroke from A to B?

(2) In how many ways can the figure (b) be drawn with a single stroke from C to D?

(a)



(b)



Q12.

Today's date is 8/31 (August 31). By merging the digits of the date (8 and 31), a number (831) is made.

(1) In how many ways can 831 be expressed as a sum of two or more sequential positive integers?

(2) Find the number of dates in a year which cannot be expressed as a sum of two or more sequential positive integers. If there is no such a date, answer as "zero" day. Note that 1/1 (January 1) makes 11, 2/10 (February 10) makes 210, and 10/1 (October 1) makes 101. Note also that, for example, the dates which make 123 are 1/23 (January 23) and 12/3 (December 3).

Q13.

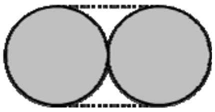
A circular cylindrical cup of inside radius r and height h is completely full of water. Find the volume $V(\alpha)$ of remaining water when it is inclined by the angle α, assuming that the bottom of the cup is under the surface of water.

Q14.

Let $I_n = \int_0^{\pi/2} \sin^n x \, dx$. Evaluate I_4 and I_5.

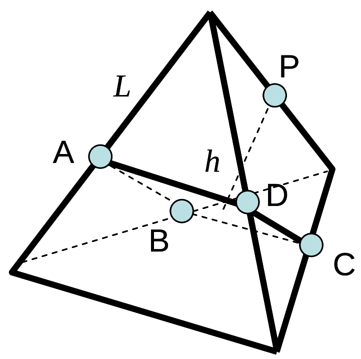
Q15.

Consider to tie cylindrical cans of radius r using a string. For example, the figure shows the case when two cans are tied. Obtain the shortest length of the string which ties seven cans.



Q16.

As shown in the figure, a square-based pyramid (P-ABCD) is made using the five midpoints on the edges of a regular tetrahedron of the edge-length L. Express the distance h between the base ABCD and the vertex P of the pyramid using L.



Q17.

Find the number of positive integers n of 1000 or less such that $n^{321}-1$ are multiples of ten.

Q18.

Two taxi companies, **A** and **B**, operate in a city, and 25% of the taxis belong to **A** while 75% belong to **B.** One day, a taxi caused an accident. One witness said that the taxi belonged to **A**. Two types of people live in the city: the honest who does not tell a lie at all and the liar who always tell a lie. 70% of people in the city are the honest while 30% are the liar. The reliability that the witness correctly identifies the company is 80%. The probability that a taxi causes an accident is the same. What is the probability that the taxi belongs to **A**?

Q19.

When the complex number z satisfies

$$z^3 = -10 + 9\sqrt{3}\, i\,,$$

obtain $z\bar{z}$ and $z + \bar{z}$, where i is the imaginary unit, and $\bar{z}$ is the complex conjugate of z.

Q20.

Initially, box **A** contains four black balls, and box **B** contains four white balls. We perform three successive ball exchanges. In each exchange, we pick simultaneously and at random a ball from each box and move it to the other box. What is the probability that at the end of the three exchanges box **A** contains two white balls?