Examinee's No.______________

Department of Nuclear Engineering and Management,
School of Engineering, The University of Tokyo
2012 Entrance Examination and Answer Sheets

Mathematics

Monday, August 29, 2011　13:00 - 15:30

Notice:

1. Do not open this booklet until the start of the examination.
2. If you find missing or badly printed pages, please raise your hand to let the supervisor know.
3. Write your examinee's number on top of this sheet. Do not write your examinee's number or your name anywhere else in this booklet.
4. Write the answer including the outline of your solution below the question.
5. Sheets for calculation are distributed separately.
6. There are 20 questions in total. Choose and answer 15 questions out of 20.
   Check the question numbers of the questions you have selected in the bottom row below. Please note that you cannot answer more than 15 questions.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

Q. 1

Calculate the integral, $\int_0^{\infty} \frac{\sin x}{x}\, dx$.

Note that the order of integration of the double integral, $\int_0^{\infty}\int_0^{\infty} e^{-xy} \sin x \, dy\, dx$, can be exchanged.

Q. 2

Solve the following differential equation about $x(t)$: $\dfrac{dx}{dt} = rx\left(1 - \dfrac{x}{K}\right)$, where r and K are constant, with the initial condition at $t = 0$: $x(0) = x_0 (\neq 0, K)$.

Q. 3

Obtain the derivative of the following function:

$$(e^{2x} + x^2 + 1)^{x^3 + \cos x}.$$

Q. 4

Three players A, B, and C play some games. In the first round, A matches B. In the second and later rounds, the winner matches the rest. If one player wins two consecutive rounds, he becomes the champion. Matches are repeated until someone wins two consecutive rounds. The probability of every player winning a match is 0.5.

(1) What is the probability for A to be the champion within 4 matches?

(2) What is the probability for A to be the champion?

Q. 5

Answer the following questions:

(1) For $0 < x < \pi$, show the formula below. n is natural number.

$$\left\{\cos\left(\frac{x}{2}\right)\right\}\left\{\cos\left(\frac{x}{2^2}\right)\right\}\cdots\left\{\cos\left(\frac{x}{2^n}\right)\right\} = \frac{\sin(x)}{2^n \sin\left(\frac{x}{2^n}\right)} .$$

(2) Evaluate the following expression.

$$\sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots$$

Q. 6

A complex function of time *t*, $s(t)$, is defined as below.

$$s(t) = \begin{cases} \exp(it^2) & |t| \le \dfrac{T}{2} \\ 0 & |t| > \dfrac{T}{2} \end{cases} .$$

Calculate the following definite integral $g(\tau)$ in the range of $\tau \ge 0$. Where $s^*(t)$ is the complex conjugate of $s(t)$, *T* is a positive constant and *i* is the imaginary unit.

$$g(\tau) = \int_{-T/2}^{T/2} s(\tau + t) s^*(t) dt \qquad (\tau \ge 0) .$$

Q. 7

Obtain the function $y(t)$ which satisfies the following differential equation, $\frac{d^2y}{dt^2} - \frac{dy}{dt} = \sin t$,

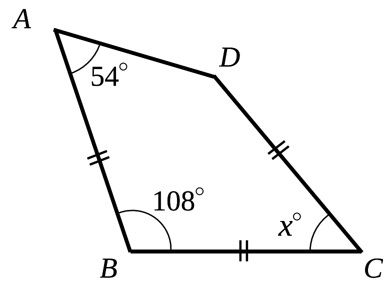
where $y(0) = 0$, $\left.\frac{dy}{dt}\right|_{t=0} = 1$.

Q. 8

There is an equilateral triangle whose side length is 1. First, we draw a circle inscribed in the equilateral triangle. Next, we draw all of the smaller circles which touch one of the drawn circles and the two sides from a vertex of the triangle. When we draw circles with repetition according to this rule infinitely, obtain the difference between the summation of all the circles' circumferences and the perimeter of the equilateral triangle. Which is longer and what is the difference between them?

Q. 9

When lengths of *AB*, *BC* and *CD* are equal, find the angle $\angle BCD$.



Q. 10

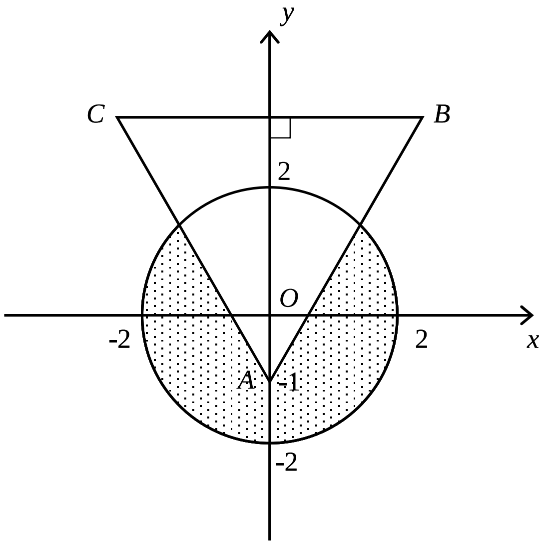
Consider a tetragon $ABCD$ inscribed in a circle, where the length of the side BC is longer than that of the side DA and the length of the side AB is longer than that of the side CD. Then, the feet of perpendiculars from the point C to the side AB, side DA, and diagonal line BD (or their extended lines) are expressed as E, F, and G, respectively. Find the angle $\angle EGF$.

Q. 11

The following ellipse can be rotated around the origin in an *x*, *y* plane: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where *a* and *b* are real numbers. Consider a rectangle that circumscribes this ellipse, where each side of the circumscribed rectangle is parallel to the *x*-axis or *y*-axis. Obtain the maximum value and the minimum value of the area *S* of the circumscribed rectangle.

Q. 12

As shown in the figure, consider the circle where the center is the origin with a radius of 2. The area of the circle is removed by the equilateral triangle *ABC* where the coordinate of *A* is (0, -1) and *y* coordinate of *B* and *C* is greater than 2. The remaining area of the circle is rotated around the *y*-axis. Obtain the volume of the rotating body.



Q. 13

There is an equilateral triangle of which the side length is $\sqrt{2}$. Draw three circles of radius 1, where the center of each circle is on each vertex of the equilateral triangle. Obtain the area of the common region of the three circles.

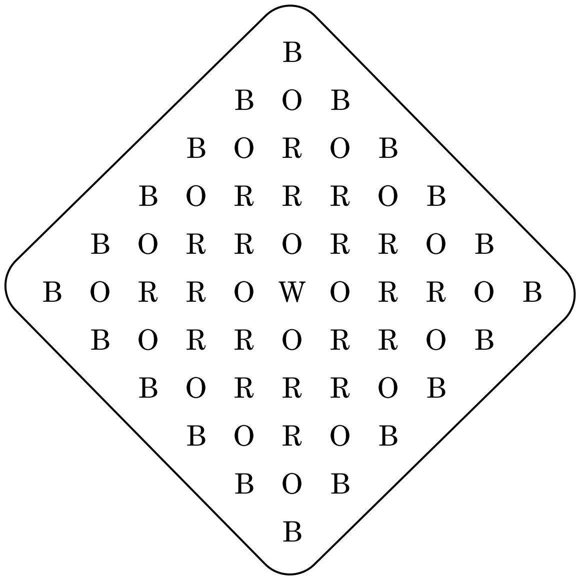
Q. 14

The following calculation shows addition in senary (6-ary) numeral system. O, N, E, T, W are digits of 1 - 5, and different characters denote different digits. Find each digit of each character.

$$\begin{array}{r} \mathrm{ONE} \\ +\ \ \mathrm{ONE} \\ \hline \mathrm{TWO} \end{array}$$

Q. 15

The following diagram shows various letters that can be used to spell out the sentence “BORROW OR ROB.” In how many different ways can you march along, from letter to letter, in the diagram and spell out this sentence? You can move in the direction of right, left, up or down from letter to letter, but cannot move in the direction of skew.



Q. 16

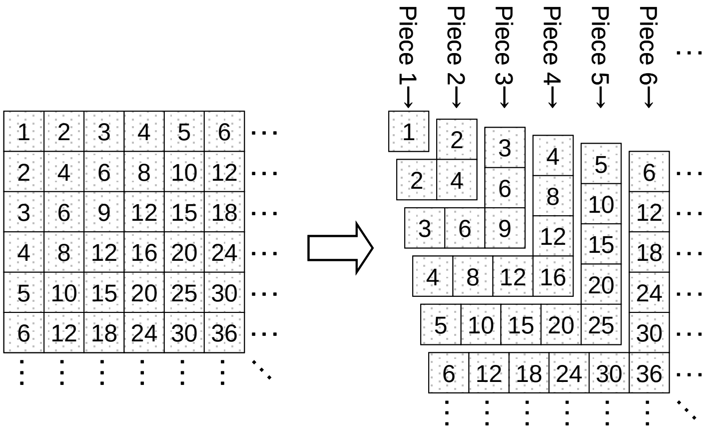
Obtain the remainder when 6^{2011} is divided by 100.

Q. 17

As shown in the following left figure, positive integers are arranged. They are divided into the pieces as shown in the right figure. The sum of the numbers in piece n is equal to n^3. Meanwhile the sum of the cubes of the first n positive integers is represented by the following equation:

$$\sum_{k=1}^{n} k^3 = an + bn^2 + cn^3 + dn^4 .$$

Find the values of a, b, c and d using the figure.



Q. 18

At first, fifty-two cards were placed facedown on a table. The first person came to the table and turned all cards up. The second person came and turned all the cards of even-numbered positions from the left side. The third person came and turned the cards of the 3rd, 6th, 9th, ⋯ positions (every third position) from the left side. The Nth person came and turned the cards of every Nth position from the left side. Finally, the 52nd person came and turned the final card on the right. After the 52nd person was gone, how many cards were placed face up on the table?

Q. 19

Obtain all pairs of natural numbers m and n that satisfy the following equation: $4n^2 + 3m - mn = 0$.

Q. 20

If $(n+2)! - n!$ is divisible by 11^6, find the smallest natural number, *n*.